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# ON THE THEORY OF ELASTIC PLATES

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**Abstract**—The isothermal infinitesimal bending theory of isotropic Cosserat surfaces with $P$ directors is developed based on the work of Green *et al.* Matrix notation presents this theory in a tractable and straightforward manner. Upon restricting the theory to initially plane surfaces, the theory uncouples into extensional and bending portions and a two dimensional matrix form of Stokes–Helmholtz decomposition theorem is applied to the bending theory. Four second-order partial differential equations in terms of four matrix stress functions are thereby obtained and all kinematic variables, stress resultants and higher order stress resultants are expressed in terms of these stress functions. This theory is applied to the pure bending of an elastic plate and a comparison is made with the three dimensional elasticity theory counterpart.

## 1. INTRODUCTION

SINCE the late 1950's there has been a revival of interest in techniques, the Cosserats [1] being among the original proponents, in which any body with a preferred orientation may be described by a position vector to a point in the body and another vector, called a director, associated with each point, which gives each point in the body six degrees of freedom. The director approach to the problem of describing an elastic continuum using two dimensional parameters was reinstigated in 1958 by Ericksen and Truesdell's [2] multiple director approach to the exact theory of rods and shells. Green, Naghdi and Wainwright [3] developed a general theory of a Cosserat surface utilizing fully consistent dynamical and thermodynamic principles of continuum mechanics and then considered nonlinear elastic, isotropic nonlinear, and isotropic linear Cosserat surfaces. The restriction contained in the Cosserat shell theory section of the previous paper, namely that of the director remaining normal to the shell surface throughout its deformation, was removed by Green and Naghdi [4]. In [5] the same authors discussed the linearized theory of an elastic Cosserat surface in relation to the theory of shells regarded as three dimensional bodies. The compatibility relations for a Cosserat surface were obtained by Crochet [6]. Naghdi [7] used Cosserat surface theory to solve the torsion of a circular cylinder while Wenner [8] derived the solution to the torsion of a cylindrical Cosserat shell. The linear theory of an elastic Cosserat plate was derived by Green and Naghdi [9] who then solved the problem of the

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bending of an infinite plate with a circular hole using this theory. Excellent review articles pertaining to the nonlinear theories of deformable surfaces are presented by Green and Naghdi [10, 11].

In several recent papers, which will be referenced extensively, Green, Laws and Naghdi [12] and Green and Naghdi [13] obtained a general nonlinear, thermodynamical theory of rods and shells using a Maclaurin series expansion for the kinematical parameters in terms of an infinite number of directors. Following their derivation of the equations of motion, with the help of invariance conditions under superposed rigid body motions, and their derivation of constitutive equations, Green, Laws and Naghdi [12] proceeded to make systematic approximations in order to recover the theory given by Green, Naghdi and Wainwright [3] who utilized one director.

In Section 2 of this paper a summary of the nonlinear and linear kinematics as well as the equations of motion of the theory of a Cosserat surface with $P$ directors is presented. The constitutive relations for an isothermal, linear, elastic plate which initially is homogeneous, free from curve and director forces, and in a state of rest, are derived from a free energy function describing a holohedrally and transversely isotropic plate in Section 3. Matrix notation is reverted to at this stage. After imposing the restrictions associated with the assumption of rectangular Cartesian coordinates in Section 4, the uncoupled field equations for the extension and bending theories are categorized prior to the application of the two dimensional Stokes–Helmholtz decomposition theorem to the transverse deflection theory. A simple and concise system of equations, governing four stress function matrices from which all stress resultants and kinematical variables are obtained, is presented for the bending theory. In Section 5, this system of equations is solved for the pure bending of an elastic plate and a comparison between the solution for the bending problem and its elastic counterpart is discussed.

Several three dimensional theories of plates are given in the literature, e.g. [14–16]. For example, A. I. Luré [14] writes a functional expression for the displacement field in terms of the two dimensional gradient operator and the thickness coordinate. This differs from the method used in the present paper where a power series expansion in terms of the thickness coordinate is employed.

## 2. KINEMATICS AND EQUATIONS OF MOTION

Points of a three dimensional continuum $b$, will be defined by a convective coordinate system $\theta^1, \theta^2, \theta^3$, and the position vector to a typical particle will be denoted by†

$$\underset{\sim}{r}^* = \underset{\sim}{r}^*(\theta^1, \theta^2, \theta^3, t). \tag{2.1}$$

The natural base vectors and their reciprocal set at points of the continuum at time $t$ are denoted by $\underset{\sim}{g}_i$ and $\underset{\sim}{g}^i$ respectively, where the range is denoted by Latin indices having the values 1, 2, 3. Whenever it is convenient $\theta^3$ will be replaced by $\xi$ and Greek indices will denote the range of values 1, 2. The notation is essentially that used in [3, 9, 12, 13, 17, 18].

The parametric equation $\xi = 0$, defines a surface $s$, in space at time $t$, whose position vector to any particle of $s$ is designated by

$$\underset{\sim}{r} = \underset{\sim}{r}(\theta^1, \theta^2, t) = \underset{\sim}{r}^*(\theta^1, \theta^2, 0, t). \tag{2.2}$$

† Three dimensional space vectors will be denoted by "$\sim$" and matrices by boldface.

The natural base vectors of the tangent plane of $s$ and their reciprocal set are denoted by $\underset{\sim}{a}_\alpha$ and $\underset{\sim}{a}^\alpha$ respectively, while the unit normal to $s$ is $\underset{\sim}{a}_3 = \underset{\sim}{a}^3$. Some of the basic geometry of the continuum $b$, and the surface $s$, which is embedded in $b$, will now be recalled, namely

$$\begin{aligned} &\underset{\sim}{g}_i = \underset{\sim}{r}^*{}_{,i}; \qquad &&\underset{\sim}{a}_\alpha = \underset{\sim}{r}_{,\alpha}, \qquad &&\underset{\sim}{a}_3 = \tfrac{1}{2}\varepsilon^{\alpha\beta}\underset{\sim}{a}_\alpha \times \underset{\sim}{a}_\beta, \\ &g_{ij} = \underset{\sim}{g}_i \cdot \underset{\sim}{g}_j; &&a_{ij} = \underset{\sim}{a}_i \cdot \underset{\sim}{a}_j, \\ &g^{ij} = \underset{\sim}{g}^i \cdot \underset{\sim}{g}^j; &&a^{ij} = \underset{\sim}{a}^i \cdot \underset{\sim}{a}^j, \\ &g = \det g_{ij}; &&a = \det a_{ij} = \det a_{\alpha\beta}, \end{aligned} \tag{2.3}$$

where $\varepsilon^{\alpha\beta} = (\sqrt{1/a})e^{\alpha\beta}$, $e^{\alpha\beta}$ is the two dimensional permutation symbol and "$_{,i}$" denotes partial differentiation with respect to $\theta^i$. The connection coefficients of $b$ and $s$ are denoted by $\Gamma^i_{jk}$ and $\bar{\Gamma}^i_{j\alpha}$ respectively. Thus

$$\underset{\sim}{g}_{i,j} = \Gamma^k_{ij}\underset{\sim}{g}_k, \qquad \underset{\sim}{a}_{i,\alpha} = \bar{\Gamma}^k_{i\alpha}\underset{\sim}{a}_k, \tag{2.4}$$

where $\bar{\Gamma}^\lambda_{\mu\alpha}$ are the Christoffel symbols of the second kind based on the metric $a_{\alpha\beta}$, $\bar{\Gamma}^3_{\mu\alpha} = b_{\mu\alpha}$ are the coefficients of the second fundamental form, $\bar{\Gamma}^\mu_{3\alpha} = -b^\mu_\alpha = -a^{\mu\lambda}b_{\lambda\alpha}$ and $\bar{\Gamma}^3_{3\alpha} = 0$. It should be noted that since the coordinate system is not normal convective, $\bar{\Gamma}^i_{j\alpha} \neq \Gamma^i_{j\alpha}(\theta^\alpha, 0, t)$. If $\underset{\sim}{v} = \underset{\sim}{v}(\theta^1, \theta^2, t)$ is any surface vector and $\underset{\sim}{v}^\alpha = \underset{\sim}{v}^\alpha(\theta^1, \theta^2, t)$ is any contravariant surface vector, then write

$$\underset{\sim}{v}_{,\alpha} = v^i{}_{;\alpha}\underset{\sim}{a}_i, \qquad \underset{\sim}{v}^\alpha|_\beta = v^{i\alpha}{}_{;\beta}\underset{\sim}{a}_i, \tag{2.5}$$

where from (2.4)

$$\begin{aligned} v^i{}_{;\alpha} &= v^i{}_{,\alpha} + \bar{\Gamma}^i_{k\alpha}v^k, \\ v^{i\alpha}{}_{;\beta} &= v^{i\alpha}{}_{,\beta} + \bar{\Gamma}^\alpha_{\gamma\beta}v^{i\gamma} + \bar{\Gamma}^i_{k\beta}v^{k\alpha}, \end{aligned} \tag{2.6}$$

and "$|_\beta$" denotes covariant differentiation with respect to the surface coordinate $\theta^\beta$, based on the metric $a_{\alpha\beta}$.

The change of base from $b$ to $s$ will be denoted by

$$\underset{\sim}{g}_i = \mu_i^{\cdot j}\underset{\sim}{a}_j, \qquad \underset{\sim}{a}_i = \mu^{\cdot j}_{\cdot i}\underset{\sim}{g}_j, \qquad \mu_i^{\cdot j}\mu^{\cdot k}_{\cdot j} = \mu_j^{\cdot k}\mu^{\cdot j}_{\cdot i} = \delta^k_i, \tag{2.7}$$

where $\mu_i^{\cdot j}$ and $\mu^{\cdot j}_{\cdot i}$, inverses of each other, are sometimes called shifters and are discussed, for example, in [17] and [18] and $\delta^k_i$ is the Kronecker delta. From (2.3) and (2.7) follows

$$g_{ij} = \mu_i^{\cdot k}\mu_j^{\cdot 1}a_{k1},$$

and hence

$$\mu = \det \mu_i^{\cdot j} = \sqrt{\left(\frac{g}{a}\right)}. \tag{2.8}$$

The shell concept may be developed by assuming that the continuum is bounded by the surfaces†

$$\xi = \alpha, \qquad \xi = \beta, \qquad (\alpha < 0 < \beta), \tag{2.9}$$

† See Ref. [12] for further restrictions on these boundary surfaces.

which are such that $s$ lies entirely between them, and the surface

$$f(\theta^1, \theta^2) = 0. \tag{2.10}$$

The position vector $\underset{\sim}{r}^*$ is now represented by a series expansion about the surface $s$, in the form

$$\underset{\sim}{r}^*(\theta^1, \theta^2, \xi, t) = \underset{\sim}{r}(\theta^1, \theta^2, t) + \sum_{N=1}^{P} \xi^N \underset{\sim}{d}_N(\theta^1, \theta^2, t), \tag{2.11}$$

where the vector functions $\underset{\sim}{d}_N$ are called the directors. It is assumed that the positive integer, $P$ is such that either the remainder of the series is zero for finite $P$ or when $P$ tends to infinity the series converges. Although the formulation is primarily that of Green, Laws and Naghdi [12], only in the case when $P$ tends to infinity the form given in [12] is recovered. The value of $P$, which is the number of directors in the expansion (2.11), is arbitrary and the question concerning the approximation when a remainder term in the Maclaurin series is needed is considered to be beyond the scope of this paper. However, it is hoped that the proposed theory may be able to shed some light on the question of approximations in the theory of plates and shells.

Some of the basic kinematical results as derived in [3] and [12] will now be recorded. From (2.3) and (2.11) it follows that

$$\begin{aligned} \underset{\sim}{g}_\alpha &= \underset{\sim}{a}_\alpha + \sum_{N=1}^{P} \xi^N \underset{\sim}{d}_{N,\alpha}, \\ \underset{\sim}{g}_3 &= \sum_{N=1}^{P} N\xi^{N-1} \underset{\sim}{d}_N. \end{aligned} \tag{2.12}$$

The shifters with the aid of (2.7) and (2.12) become

$$\mu_\alpha^{.j} = \delta_\alpha^j + \sum_{N=1}^{P} \xi^N d_N^j{}_{;\alpha}, \qquad \mu_3^{.j} = \sum_{N=1}^{P} N\xi^{N-1} d_N^j, \tag{2.13}$$

where, from (2.6)

$$\underset{\sim}{d}_N = d_N^i \underset{\sim}{a}_i, \qquad \underset{\sim}{d}_{N,\alpha} = d_N^i{}_{;\alpha} \underset{\sim}{a}_i, \tag{2.14}$$

and

$$\begin{aligned} d_N^\nu{}_{;\alpha} &= d_N^\nu|_\alpha - b_\alpha^\nu d_N^3 \equiv \lambda_N{}^\nu_{.\alpha}, \\ d_N^3{}_{;\alpha} &= d_N^3{}_{,\alpha} + b_{\mu\alpha} d_N^\mu \equiv \lambda_N{}^3_{.\alpha}. \end{aligned} \tag{2.15}$$

In the undeformed configuration the position vectors to a typical particle, denoted by $\underset{\sim}{R}^*(\theta^1, \theta^2, \xi)$, and to the particle on the surface $S$, obtained from the typical particle's position by setting $\xi = 0$, denoted by $\underset{\sim}{R}(\theta^1, \theta^2)$, are related through the expansion (2.11) evaluated at time equal to zero. Thus

$$\underset{\sim}{R}^*(\theta^1, \theta^2, \xi) = \underset{\sim}{R}(\theta^1, \theta^2) + \sum_{N=1}^{P} \xi^N \underset{\sim}{D}_N(\theta^1, \theta^2), \tag{2.16}$$

where $\underset{\sim}{D}_N$ are the initial undeformed directors. Similar expressions to those contained in (2.3)–(2.8) and (2.12)–(2.15), hold in the undeformed configuration with majuscules replacing their miniscule counterparts, $\mathring{\mu}_i^{\cdot j}$ replacing $\mu_i^{\cdot j}$, $\mathring{\bar{\Gamma}}{}^i_{j\alpha}$ replacing $\bar{\Gamma}^i_{j\alpha}$ and "$_{:\alpha}$" replacing "$_{;\alpha}$".

The position vectors in the deformed and undeformed configurations are related to the displacement vectors by

$$\mathbf{r}^* = \mathbf{R}^* + \mathbf{U}^*, \qquad \mathbf{r} = \mathbf{R} + \mathbf{U}, \tag{2.17}$$

where

$$\mathbf{U}^* = U^{*i}\mathbf{G}_i, \qquad \mathbf{U} = U^i\mathbf{A}_i, \tag{2.18}$$

are the three dimensional and surface displacements respectively. From (2.11), (2.16), (2.17) and (2.18)

$$\mathbf{U}^* = \mathbf{U} + \sum_{N=1}^{P} \xi^N \bar{\boldsymbol{\delta}}_N, \tag{2.19}$$

or in component form

$$U^{*i}\mathring{\mu}_i^{\cdot j} = U^j + \sum_{N=1}^{P} \xi^N \bar{\delta}^j_N, \tag{2.20}$$

where

$$\bar{\boldsymbol{\delta}}_N = \bar{\delta}^j_N \mathbf{A}_j = \mathbf{d}_N - \mathbf{D}_N, \qquad \bar{\delta}^j_N = d^i_N \mathbf{a}_i \cdot \mathbf{A}^j - D^j_N. \tag{2.21}$$

As in [3] the extended definitions

$$\delta_{Ni} = d_{Ni} - D_{Ni}, \tag{2.22}$$

$$\varkappa_{Ni\alpha} = \mathbf{a}_i \cdot \mathbf{d}_{N,\alpha} - \mathbf{A}_i \cdot \mathbf{D}_{N,\alpha} = d_{Ni;\alpha} - D_{Ni:\alpha}, \tag{2.23}$$

as well as

$$2e_{\alpha\beta} = a_{\alpha\beta} - A_{\alpha\beta}, \tag{2.24}$$

are introduced. It is of interest to note that although $\delta_{Ni}$ transforms as the components of a covariant vector, it is not associated with a surface vector.

Attention is now restricted to the kinematical theory of infinitesimal displacements. The displacement of the surface $s$ and the director displacements are given by $(2.18)_2$ and (2.21), respectively. Upon linearization of (2.21) through (2.24) in a manner similar to Green, Naghdi and Wainwright [3] the following linearized kinematical results are obtained

$$\begin{aligned}\varkappa_{N\beta\alpha} &= \bar{\boldsymbol{\delta}}_{N,\alpha} \cdot \mathbf{A}_\beta + \mathbf{U}_{,\beta} \cdot \mathbf{D}_{N,\alpha} \\ &= \bar{\delta}_{N\beta:\alpha} + U_{i:\beta} D^i_{N:\alpha},\end{aligned} \tag{2.25}$$

$$\begin{aligned}\varkappa_{N3\alpha} &= \bar{\boldsymbol{\delta}}_{N,\alpha} \cdot \mathbf{A}_3 - U_{3:\gamma}\mathbf{A}^\gamma \cdot \mathbf{D}_{N,\alpha} \\ &= \bar{\delta}_{N3:\alpha} - U_{3:\gamma} D^\gamma_{N:\alpha},\end{aligned} \tag{2.26}$$

$$\begin{aligned}2e_{\alpha\beta} &= \mathbf{A}_\alpha \cdot \mathbf{U}_{,\beta} + \mathbf{A}_\beta \cdot \mathbf{U}_{,\alpha} \\ &= U_{\alpha:\beta} + U_{\beta:\alpha}.\end{aligned} \tag{2.27}$$

In addition the two forms of the director displacement $\bar{\delta}_{Ni}$ and $\delta_{Ni}$ are related through

$$\begin{aligned}\bar{\delta}_{N\alpha} &= \delta_{N\alpha} - (U^\beta{}_{:\alpha} D_{N\beta} + U_{3:\alpha} D_{N3}), \\ \bar{\delta}_{N3} &= \delta_{N3} + U_3{}^{:\beta} D_{N\beta}.\end{aligned} \tag{2.28}$$

Expressions for the three dimensional strains in terms of surface and director displacement gradients may be derived utilizing (2.19) provided that the differentiated series either exhibits equality for finite $P$ or converges uniformly when $P$ tends to infinity. The continuum's linear strain components are given by

$$2\gamma_{ij} = (\underset{\sim}{G}_i \cdot \underset{\sim}{U}^*_{,j} + \underset{\sim}{G}_j \cdot \underset{\sim}{U}^*_{,i}), \tag{2.29}$$

which upon using (2.5), (2.7), (2.17), (2.19) and (2.29), become

$$2\gamma_{\alpha\beta} = \mathring{\mu}_\alpha^{\cdot k}\left(U_k{:}_\beta + \sum_{N=1}^{P} \xi^N \bar{\delta}_{Nk}{:}_\beta\right) + \mathring{\mu}_\beta^{\cdot k}\left(U_k{:}_\alpha + \sum_{N=1}^{P} \xi^N \bar{\delta}_{Nk}{:}_\alpha\right), \tag{2.30}$$

$$2\gamma_{\alpha 3} = \mathring{\mu}_\alpha^{\cdot k} \sum_{N=1}^{P} N\xi^{N-1}\bar{\delta}_{Nk} + \mathring{\mu}_3^{\cdot k}\left(U_k{:}_\alpha + \sum_{N=1}^{P} \xi^N \bar{\delta}_{Nk}{:}_\alpha\right), \tag{2.31}$$

$$\gamma_{33} = \mathring{\mu}_3^{\cdot k} \sum_{N=1}^{P} N\xi^{N-1}\bar{\delta}_{Nk}. \tag{2.32}$$

A simplification of the kinematics occurs when the following identification of the initial directors is made

$$\underset{\sim}{D}_1 = \underset{\sim}{A}_3, \qquad \underset{\sim}{D}_N = \underset{\sim}{0} \quad \text{for } N > 1. \tag{2.33}$$

This choice corresponds to describing the initial configuration by families of parallel surfaces. Thus the convective coordinates $\theta^i$ are normal coordinates in the initial configuration and the shell is of uniform thickness in this configuration. With this interpretation the undeformed configuration's shifters from (2.13) and (2.33), become

$$\begin{aligned} \mathring{\mu}_\alpha^{\cdot\beta} &= \delta_\alpha^\beta - B_\alpha^\beta \xi, \qquad & \mathring{\mu}_\alpha^{\cdot 3} &= 0 \\ \mathring{\mu}_3^{\cdot\alpha} &= 0, & \mathring{\mu}_3^{\cdot 3} &= 1 \end{aligned} \tag{2.34}$$

which, along with (2.33), reduce the linear kinematical relations (2.20), (2.25)–(2.28) to

$$U^{*\beta} - U^{*\alpha}B_\alpha^\beta \xi = U^\beta + \sum_{N=1}^{P} \xi^N \bar{\delta}_N^\beta, \qquad U^{*3} = U^3 + \sum_{N=1}^{P} \xi^N \bar{\delta}_N^3; \tag{2.35}$$

$$\bar{\delta}_{1\alpha} = \delta_{1\alpha} - U_3{:}_\alpha, \qquad \bar{\delta}_{N\alpha} = \delta_{N\alpha} \quad \text{for } N > 1, \qquad \bar{\delta}_{N3} = \delta_{N3}; \tag{2.36}$$

$$\varkappa_{1\beta\alpha} = \bar{\delta}_{1\beta}{:}_\alpha - U^\gamma{:}_\beta B_{\gamma\alpha}, \qquad \varkappa_{N\beta\alpha} = \delta_{N\beta}{:}_\alpha \quad \text{for } N > 1, \tag{2.37}$$

$$\varkappa_{13\alpha} = \bar{\delta}_{13}{:}_\alpha + U_3{:}^\gamma B_{\gamma\alpha}, \qquad \varkappa_{N3\alpha} = \delta_{N3}{:}_\alpha \quad \text{for } N > 1; \tag{2.38}$$

Three dimensional strains (2.30)–(2.32), are also reduced by (2.33) and (2.34) to

$$\begin{aligned} 2\gamma_{\alpha\beta} &= 2e_{\alpha\beta} + \sum_{N=1}^{P} \xi^N(\bar{\delta}_{N\alpha}{:}_\beta + \bar{\delta}_{N\beta}{:}_\alpha) \\ &\qquad - \xi(B_\alpha^\gamma U_\gamma{:}_\beta + B_\beta^\gamma U_\gamma{:}_\alpha) - \sum_{N=1}^{P} \xi^{N+1}(B_\alpha^\gamma \bar{\delta}_{N\gamma}{:}_\beta + B_\beta^\gamma \bar{\delta}_{N\gamma}{:}_\alpha), \\ 2\gamma_{\alpha 3} &= U_3{:}_\alpha + \sum_{N=1}^{P} (\xi^N \bar{\delta}_{N3}{:}_\alpha + N\xi^{N-1}\bar{\delta}_{N\alpha} - NB_\alpha^\gamma \xi^N \bar{\delta}_{N\gamma}), \\ \gamma_{33} &= \sum_{N=1}^{P} N\xi^{N-1}\delta_{N3}. \end{aligned} \tag{2.39}$$

Returning to the consideration of the nonlinear theory, the equations of motion for the shell may be taken as those given in [12, 13]. The same form of these equations may also be obtained by integrating the equations of motion across the thickness, both prior to and following multiplication by powers of the thickness coordinate. In order to record these equations for future use the following definitions and notations are required. Let $\underset{\sim}{v}$ be the outward unit normal in the surface $s$, to a curve of the form (2.10) with $\xi = 0$, and $\underset{\sim}{t}^i$ be the stress vectors for each coordinate surface per unit area of the deformed body, with shifted stress tensor components given by

$$\underset{\sim}{t}^i = \sigma^{ji}\underset{\sim}{a}_j, \tag{2.40}$$

where $\sigma^{ji}$ is, in general, a nonsymmetric tensor related to the three dimensional symmetric contravariant stress tensor† $t^{ji}$, through

$$\sigma^{ji} = t^{ki}\mu_k^{\cdot j}. \tag{2.41}$$

The definitions of stress resultants are given by

$$\underset{\sim}{N}^\alpha = N^{i\alpha}\underset{\sim}{a}_i = \int_\alpha^\beta \mu\underset{\sim}{t}^\alpha\, d\xi, \qquad \underset{\sim}{N} = \underset{\sim}{N}^\alpha v_\alpha, \tag{2.42}$$

$$\underset{\sim}{M}_N^\alpha = M_N^{i\alpha}\underset{\sim}{a}_i = \int_\alpha^\beta \mu\xi^N\underset{\sim}{t}^\alpha\, d\xi, \qquad \underset{\sim}{M}_N = \underset{\sim}{M}_N^\alpha v_\alpha, \tag{2.43}$$

$$\underset{\sim}{m}_N = m_N^i\underset{\sim}{a}_i = \int_\alpha^\beta \mu N\xi^{N-1}\underset{\sim}{t}^3\, d\xi, \tag{2.44}$$

$$\rho\underset{\sim}{F} = \rho F^i\underset{\sim}{a}_i = \int_\alpha^\beta \rho^*\mu(\underset{\sim}{f}^* - \underset{\sim}{c}^*)\, d\xi + \underset{\sim}{p}, \tag{2.45}$$

$$\rho\underset{\sim}{L}_N = \rho L_N^i\underset{\sim}{a}_i = \int_\alpha^\beta \rho^*\mu\xi^N(\underset{\sim}{f}^* - \underset{\sim}{c}^*)\, d\xi + \underset{\sim}{l}_N, \tag{2.46}$$

where

$$\underset{\sim}{p} = p^i\underset{\sim}{a}_i = \mu\underset{\sim}{t}^3\Big|_{\xi=\alpha}^{\xi=\beta}, \tag{2.47}$$

$$\underset{\sim}{l}_N = l_N^i\underset{\sim}{a}_i = \mu\xi^N\underset{\sim}{t}^3\Big|_{\xi=\alpha}^{\xi=\beta}. \tag{2.48}$$

In the preceding equations, $\underset{\sim}{f}^*$ is the body force vector, $\underset{\sim}{c}^*$ is the acceleration which contains both the surface and all director accelerations and $\rho^*$ is the density.

Thus, in terms of the definitions (2.42)–(2.48) the equations of motion for the shell may be written as

$$\underset{\sim}{N}^\alpha|_\alpha + \rho\underset{\sim}{F} = \underset{\sim}{0}, \tag{2.49}$$

$$\underset{\sim}{M}_N^\alpha|_\alpha + \rho\underset{\sim}{L}_N - \underset{\sim}{m}_N = \underset{\sim}{0}, \tag{2.50}$$

† This interpretation of $t^{ji}$ agrees with Truesdell and Toupin's [19], Section 203, but not with Green *et al.*'s.

and

$$\underset{\sim}{a}_\alpha \times \underset{\sim}{N}^\alpha + \sum_{N=1}^{P} \underset{\sim}{d}_{N,\alpha} \times \underset{\sim}{M}_N^\alpha + \sum_{N=1}^{P} \underset{\sim}{d}_N \times \underset{\sim}{m}_N = \underset{\sim}{0}. \tag{2.51}$$

The scalar forms of the equations of motion (2.49) and (2.50) utilizing (2.5) and (2.42) to (2.46) are

$$N^{i\alpha}{}_{;\alpha} + \rho F^i = 0. \tag{2.52}$$

$$M_N^{i\alpha}{}_{;\alpha} + \rho L_N^i - m_N^i = 0, \tag{2.53}$$

while the scalar form of (2.51), using (2.14), (2.42), (2.43) and (2.44) along with the properties of cross-products between surface base vectors becomes

$$\varepsilon_{\alpha\beta} N'^{\beta\alpha} = 0, \quad \text{or} \quad N'^{\beta\alpha} = N'^{\alpha\beta}, \tag{2.54}$$

and

$$N^{3\alpha} + \sum_{N=1}^{P} (d_N^\alpha m_N^3 - d_N^3 m_N^\alpha) + \sum_{N=1}^{P} (d_N^\alpha{}_{;\gamma} M_N^{3\gamma} - d_N^3{}_{;\gamma} M_N^{\alpha\gamma}) = 0, \tag{2.55}$$

where

$$N'^{\beta\alpha} = N^{\beta\alpha} - \sum_{N=1}^{P} (d_N^\beta{}_{;\gamma} M_N^{\alpha\gamma} + d_N^\beta m_N^\alpha). \tag{2.56}$$

## 3. CONSTITUTIVE EQUATIONS FOR A LINEAR ISOTROPIC ELASTIC PLATE

Attention is now restricted to the isothermal infinitesimal deformations of an elastic Cosserat plate with $P$ directors, which initially is homogeneous, free from all curve and director forces and in a state of rest. However, the constitutive equations to be derived in this section may be applied to thin shells if all terms of $O(h/R)$, $h$ being the thickness and $R$ the smallest radius of curvature, are neglected. Further discussions of this property have been given by Green and Naghdi [5]. With initial directors as assumed in equation (2.33), the Helmholtz free energy should include terms involving $D_{1\alpha;\beta}$ which is the negative of the initial coefficients of the second fundamental form $B_{\beta\alpha}$. If, however, the initial surface $s$ is assumed to be a plane, the value of all kinematical variables at time $t = 0$ is zero. For the above reasons it is sufficient for the Helmholtz free energy per unit mass $\rho_0 A$, to be expressed as a quadratic function of $e_{\alpha\beta}$, $\varkappa_{Ni\alpha}$ and $\delta_{Ni}$, where $\rho_0$ is the density per unit area of the initially undeformed plate. Hence the free energy

$$\rho_0 A = \rho_0 A(e_{\alpha\beta}, \varkappa_{Ni\alpha}, \delta_{Ni}; A^{\alpha\beta}), \tag{3.1}$$

may be written as

$$
\begin{aligned}
\rho_0 A = {}& {}_1C^{\alpha\beta\gamma\delta} e_{\alpha\beta} e_{\gamma\delta} + \sum_{N,M=1}^{P} {}_2C_{NM}^{\alpha\beta\gamma\delta} \varkappa_{N\alpha\beta} \varkappa_{M\gamma\delta} + \sum_{N=1}^{P} {}_3C_N^{\alpha\beta\gamma\delta} e_{\alpha\beta} \varkappa_{N\gamma\delta} \\
& + \sum_{N,M=1}^{P} {}_1C_{NM}^{\alpha\beta\gamma} \varkappa_{N3\alpha} \varkappa_{M\beta\gamma} + \sum_{N=1}^{P} {}_2C_N^{\alpha\beta\gamma} e_{\alpha\beta} \delta_{N\gamma} + \sum_{N=1}^{P} {}_3C_N^{\alpha\beta\gamma} e_{\alpha\beta} \varkappa_{N3\gamma} \\
& + \sum_{N,M=1}^{P} {}_4C_{NM}^{\alpha\beta\gamma} \delta_{N\alpha} \varkappa_{M\beta\gamma} + \sum_{N,M=1}^{P} {}_1C_{NM}^{\alpha\beta} \delta_{N\alpha} \delta_{M\beta} + \sum_{N,M=1}^{P} {}_2C_{NM}^{\alpha\beta} \varkappa_{N3\alpha} \varkappa_{M3\beta} \\
& + \sum_{N,M=1}^{P} {}_3C_{NM}^{\alpha\beta} \delta_{N\alpha} \varkappa_{M3\beta} + \sum_{N=1}^{P} {}_4C_N^{\alpha\beta} e_{\alpha\beta} \delta_{N3} + \sum_{N,M=1}^{P} {}_5C_{NM}^{\alpha\beta} \varkappa_{N\alpha\beta} \delta_{M3} \\
& + \sum_{N,M=1}^{P} {}_1C_{NM}^{\alpha} \delta_{N\alpha} \delta_{M3} + \sum_{N,M=1}^{P} {}_2C_{NM}^{\alpha} \varkappa_{N3\alpha} \delta_{M3} + \sum_{N,M=1}^{P} C_{NM} \delta_{N3} \delta_{M3},
\end{aligned} \tag{3.2}
$$

where the coefficients are constants which satisfy symmetry conditions similar to those given by Green, Naghdi and Wainwright [3]. The plate is now assumed to be isotropic with a center of symmetry and thus all coefficients of odd order must vanish. In addition, remaining coefficients must be homogeneous, linear functions of products of $A^{\alpha\beta}$. In order to imitate the symmetries associated with a plate which is transversely isotropic with respect to the normal to the surface $S$, the free energy must remain invariant under the transformations

$$
\begin{aligned}
\delta_{N\alpha} &\to (-1)^N \delta_{N\alpha}, \\
\delta_{N3} &\to (-1)^{N+1} \delta_{N3}, \\
U_3 &\to -U_3.
\end{aligned} \tag{3.3}
$$

For ease in the further development of this theory, matrix notation is now introduced and, in what follows, matrices will be designated by boldface. The kinematical variables $\delta_{Ni}, \varkappa_{Ni\alpha}$ are now decomposed into $P/2 \times 1$ matrices, where $P$ is an even integer, in the following manner. Kinematic quantities which involve the odd values of $\delta_{N\alpha}$ and even values of $\delta_{N3}$ will be designated simply by $\boldsymbol{\delta}_i$ and $\boldsymbol{\varkappa}_{i\alpha}$; that is

$$
\begin{aligned}
\boldsymbol{\delta}_\alpha^T &= (\delta_{1\alpha}, \delta_{3\alpha}, \delta_{5\alpha}, \dots, \delta_{(P-1)\alpha}), \\
\boldsymbol{\delta}_3^T &= (\delta_{23}, \delta_{43}, \delta_{63}, \dots, \delta_{P3}), \\
\boldsymbol{\varkappa}_{\alpha\beta}^T &= (\varkappa_{1\alpha\beta}, \varkappa_{3\alpha\beta}, \varkappa_{5\alpha\beta}, \dots, \varkappa_{(P-1)\alpha\beta}), \\
\boldsymbol{\varkappa}_{3\alpha}^T &= (\varkappa_{23\alpha}, \varkappa_{43\alpha}, \varkappa_{63\alpha}, \dots, \varkappa_{P3\alpha}),
\end{aligned} \tag{3.4}
$$

and kinematic quantities which involve the even values of $\delta_{N\alpha}$ and odd values of $\delta_{N3}$ will be denoted by $\tilde{\boldsymbol{\delta}}_i$ and $\tilde{\boldsymbol{\varkappa}}_{i\alpha}$; namely

$$
\begin{aligned}
\tilde{\boldsymbol{\delta}}_\alpha^T &= (\delta_{2\alpha}, \delta_{4\alpha}, \delta_{6\alpha}, \dots, \delta_{P\alpha}), \\
\tilde{\boldsymbol{\delta}}_3^T &= (\delta_{13}, \delta_{33}, \delta_{53}, \dots, \delta_{(P-1)3}), \\
\tilde{\boldsymbol{\varkappa}}_{\alpha\beta}^T &= (\varkappa_{2\alpha\beta}, \varkappa_{4\alpha\beta}, \varkappa_{6\alpha\beta}, \dots, \varkappa_{P\alpha\beta}), \\
\tilde{\boldsymbol{\varkappa}}_{3\alpha}^T &= (\varkappa_{13\alpha}, \varkappa_{33\alpha}, \varkappa_{53\alpha}, \dots, \varkappa_{(P-1)3\alpha}).
\end{aligned} \tag{3.5}
$$

In the above superscript $T$ indicates the transpose and $P$ in (2.11) is restricted to an even integer so that $\boldsymbol{\delta}_i$ and $\tilde{\boldsymbol{\delta}}_i$ have the same number of elements. However, only a slight modification is needed in what follows if one wishes to consider $P$ as an odd integer. Imposing the restriction (3.3), as well as the condition of holohedral isotropy, on (3.2) the algebraic expression for the free energy may be written as the sum of two terms: one which represents the transverse deflection or bending theory $\rho_0 A'$, involving the odd values of $\delta_{N\alpha}$ and even values of $\delta_{N3}$, and the other portion which represents the generalized plane stress or extensional theory $\rho_0\tilde{A}$, involving the even values of $\delta_{N\alpha}$ and odd values of $\delta_{N3}$. Thus the free energy may be expressed as

$$\rho_0 A = \rho_0 A' + \rho_0 \tilde{A}, \tag{3.6}$$

where

$$\begin{aligned}\rho_0 A' = {} & \tfrac{1}{2}A^{\alpha\beta}\boldsymbol{\delta}_\alpha^T\boldsymbol{\alpha}_3\boldsymbol{\delta}_\beta + \tfrac{1}{2}\boldsymbol{\delta}_3^T\boldsymbol{\alpha}_4\boldsymbol{\delta}_3 \\ & + \tfrac{1}{2}\boldsymbol{\varkappa}_{\alpha\beta}^T[\boldsymbol{\alpha}_5 A^{\alpha\beta}A^{\gamma\delta} + \boldsymbol{\alpha}_6 A^{\alpha\gamma}A^{\beta\delta} + \boldsymbol{\alpha}_7 A^{\alpha\delta}A^{\beta\gamma}]\boldsymbol{\varkappa}_{\gamma\delta} \\ & + \tfrac{1}{2}A^{\alpha\beta}\boldsymbol{\varkappa}_{3\alpha}^T\boldsymbol{\alpha}_8\boldsymbol{\varkappa}_{3\beta} + A^{\alpha\beta}\boldsymbol{\varkappa}_{\alpha\beta}^T\boldsymbol{\alpha}_{12}\boldsymbol{\delta}_3 + A^{\alpha\beta}\boldsymbol{\delta}_\alpha^T\boldsymbol{\alpha}_{13}\boldsymbol{\varkappa}_{3\beta},\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\rho_0\tilde{A} = {} & \tfrac{1}{2}[\alpha_1 A^{\alpha\beta}A^{\gamma\delta} + \alpha_2(A^{\alpha\gamma}A^{\beta\delta} + A^{\alpha\delta}A^{\beta\gamma})]e_{\alpha\beta}e_{\gamma\delta} + \tfrac{1}{2}A^{\alpha\beta}\tilde{\boldsymbol{\delta}}_\alpha^T\boldsymbol{\beta}_3\tilde{\boldsymbol{\delta}}_\beta + \tfrac{1}{2}\tilde{\boldsymbol{\delta}}_3^T\boldsymbol{\beta}_4\tilde{\boldsymbol{\delta}}_3 \\ & + \tfrac{1}{2}\tilde{\boldsymbol{\varkappa}}_{\alpha\beta}^T[\boldsymbol{\beta}_5 A^{\alpha\beta}A^{\gamma\delta} + \boldsymbol{\beta}_6 A^{\alpha\gamma}A^{\beta\delta} + \boldsymbol{\beta}_7 A^{\alpha\delta}A^{\beta\gamma}]\tilde{\boldsymbol{\varkappa}}_{\gamma\delta} + \tfrac{1}{2}A^{\alpha\beta}\tilde{\boldsymbol{\varkappa}}_{3\alpha}^T\boldsymbol{\beta}_8\tilde{\boldsymbol{\varkappa}}_{3\beta} + A^{\alpha\beta}e_{\alpha\beta}\boldsymbol{\beta}_9^T\tilde{\boldsymbol{\delta}}_3 \\ & + e_{\alpha\beta}[\boldsymbol{\beta}_{10}^T A^{\alpha\beta}A^{\gamma\delta} + \boldsymbol{\beta}_{11}^T(A^{\alpha\gamma}A^{\beta\delta} + A^{\alpha\delta}A^{\beta\gamma})]\tilde{\boldsymbol{\varkappa}}_{\gamma\delta} + A^{\alpha\beta}\tilde{\boldsymbol{\varkappa}}_{\alpha\beta}^T\boldsymbol{\beta}_{12}\tilde{\boldsymbol{\delta}}_3 \\ & + A^{\alpha\beta}\tilde{\boldsymbol{\delta}}_\alpha^T\boldsymbol{\beta}_{13}\tilde{\boldsymbol{\varkappa}}_{3\beta}.\end{aligned} \tag{3.8}$$

The material coefficient matrices $\boldsymbol{\beta}_9, \boldsymbol{\beta}_{10}, \boldsymbol{\beta}_{11}$ are $P/2 \times 1$ matrices, $\boldsymbol{\alpha}_{12}^T = \boldsymbol{\beta}_{12}, \boldsymbol{\alpha}_{13}^T = \boldsymbol{\beta}_{13}$ and all other material coefficients are $P/2 \times P/2$ symmetric matrices. Since the free energy is positive definite these material coefficient matrices have further restrictions. It is easily shown that the inverse of certain material coefficient matrices, which are used, exist.

The bending and extensional constitutive equations may be derived from (3.6), (3.7) and (3.8) by using partial derivatives of the free energy equivalent to those given by Green, Laws and Naghdi [12]. Use will be made of the notation exemplified by $\partial_{\boldsymbol{\delta}\alpha}A'$ as indicating a $P/2 \times 1$ matrix obtained by taking the partial derivative of the free energy with respect to each $\delta_{N\alpha}$, $N = 1, 3, 5, \ldots, P-1$. Thus the constitutive relations are given by†

$$\mathbf{m}^\alpha = \rho_0\partial_{\boldsymbol{\delta}_\alpha}A' = A^{\alpha\beta}\boldsymbol{\alpha}_3\boldsymbol{\delta}_\beta + A^{\alpha\beta}\boldsymbol{\alpha}_{13}\boldsymbol{\varkappa}_{3\beta}, \tag{3.9}$$

$$\mathbf{m}^3 = \rho_0\partial_{\boldsymbol{\delta}_3}A' = \boldsymbol{\alpha}_4\boldsymbol{\delta}_3 + A^{\alpha\beta}\boldsymbol{\alpha}_{12}^T\boldsymbol{\varkappa}_{\alpha\beta}, \tag{3.10}$$

$$\mathbf{M}^{\alpha\beta} = \rho_0\partial_{\boldsymbol{\varkappa}_{\alpha\beta}}A' = [\boldsymbol{\alpha}_5 A^{\alpha\beta}A^{\gamma\delta} + \boldsymbol{\alpha}_6 A^{\alpha\gamma}A^{\beta\delta} + \boldsymbol{\alpha}_7 A^{\alpha\delta}A^{\beta\gamma}]\boldsymbol{\varkappa}_{\gamma\delta} + A^{\alpha\beta}\boldsymbol{\alpha}_{12}\boldsymbol{\delta}_3, \tag{3.11}$$

$$\mathbf{M}^{3\alpha} = \rho_0\partial_{\boldsymbol{\varkappa}_{3\alpha}}A' = A^{\alpha\beta}\boldsymbol{\alpha}_8\boldsymbol{\varkappa}_{3\beta} + A^{\beta\alpha}\boldsymbol{\alpha}_{13}^T\boldsymbol{\delta}_\beta; \tag{3.12}$$

and

$$\begin{aligned}N'^{\beta\alpha} = \rho_0\partial_{e_{\alpha\beta}}\tilde{A} = {} & [\alpha_1 A^{\alpha\beta}A^{\gamma\delta} + \alpha_2(A^{\alpha\gamma}A^{\beta\delta} + A^{\alpha\delta}A^{\beta\gamma})]e_{\gamma\delta} \\ & + A^{\alpha\beta}\boldsymbol{\beta}_9^T\tilde{\boldsymbol{\delta}}_3 + [\boldsymbol{\beta}_{10}^T A^{\alpha\beta}A^{\gamma\delta} + \boldsymbol{\beta}_{11}^T(A^{\alpha\gamma}A^{\beta\delta} + A^{\alpha\delta}A^{\beta\gamma})]\tilde{\boldsymbol{\varkappa}}_{\gamma\delta},\end{aligned} \tag{3.13}$$

† By the notation $\partial_{e_{\alpha\beta}}$ it is understood to mean $\frac{1}{2}(\partial_{e_{\alpha\beta}} + \partial_{e_{\beta\alpha}})$, since $e_{\alpha\beta}$ is symmetric.

$$\tilde{\mathbf{m}}^{\alpha} = \rho_0 \partial_{\bar{\mathbf{\delta}}_\alpha} \tilde{A} = A^{\alpha\beta}\boldsymbol{\beta}_3\bar{\boldsymbol{\delta}}_\beta + A^{\alpha\beta}\boldsymbol{\beta}_{13}\bar{\boldsymbol{\varkappa}}_{3\beta}, \tag{3.14}$$

$$\tilde{\mathbf{m}}^{3} = \rho_0 \partial_{\bar{\mathbf{\delta}}_3} \tilde{A} = \boldsymbol{\beta}_4\bar{\boldsymbol{\delta}}_3 + A^{\alpha\beta}e_{\alpha\beta}\boldsymbol{\beta}_9 + A^{\alpha\beta}\boldsymbol{\beta}_{12}^T\bar{\boldsymbol{\varkappa}}_{\alpha\beta}, \tag{3.15}$$

$$\tilde{\mathbf{M}}^{\alpha\beta} = \rho_0 \partial_{\bar{\mathbf{\varkappa}}_{\alpha\beta}} \tilde{A} = [\boldsymbol{\beta}_5 A^{\alpha\beta}A^{\gamma\delta} + \boldsymbol{\beta}_6 A^{\alpha\gamma}A^{\beta\delta} + \boldsymbol{\beta}_7 A^{\alpha\delta}A^{\beta\gamma}]\bar{\boldsymbol{\varkappa}}_{\gamma\delta} + [\boldsymbol{\beta}_{10}A^{\alpha\beta}A^{\gamma\delta}$$
$$+\boldsymbol{\beta}_{11}(A^{\gamma\alpha}A^{\delta\beta} + A^{\gamma\beta}A^{\delta\alpha})]e_{\gamma\delta} + A^{\alpha\beta}\boldsymbol{\beta}_{12}\bar{\boldsymbol{\delta}}_3, \tag{3.16}$$

$$\tilde{\mathbf{M}}^{3\alpha} = \rho_0 \partial_{\bar{\mathbf{\varkappa}}_{3\alpha}} \tilde{A} = A^{\alpha\beta}\boldsymbol{\beta}_8\bar{\boldsymbol{\varkappa}}_{3\beta} + A^{\beta\alpha}\boldsymbol{\beta}_{13}^T\bar{\boldsymbol{\delta}}_\beta. \tag{3.17}$$

The stress resultant $P/2\times 1$ matrices in (3.9)–(3.17), recalling the renumbering given by (3.4) and (3.5), have terms consistent with the interpretation of the kinematical matrix with which each partial derivative of the free energy was taken. It should be noted that because of the choice of initial directors and since a plane surface is being considered, the equations of motion (2.54) and (2.56) specify that

$$N^{\alpha\beta} = N'^{\alpha\beta} = N'^{\beta\alpha} = N^{\beta\alpha}, \tag{3.18}$$

and hence the constitutive equation (3.13) may be utilized to describe the symmetric $N^{\alpha\beta}$.

## 4. DECOMPOSITION OF BENDING THEORY

The basic equations of the linear theory of an elastic Cosserat plate with $P$ directors under static loading are now recorded in matrix notation. The kinematical relations for a plate will now be referred to rectangular Cartesian coordinates and are obtained from the equations contained in Section 2 by setting

$$A^{\alpha\beta} = \delta_{\alpha\beta}, \qquad \mathring{\Gamma}^{\lambda}_{\mu\alpha} = 0, \qquad B^{\lambda}_{\mu} = 0. \tag{4.1}$$

Using standard procedures, for example by replacing $\underset{\sim}{d}_N$ and $b^{\lambda}_{\alpha}$ to the first order with $\underset{\sim}{D}_N$ and $B^{\lambda}_{\alpha}$ respectively, the equations of motion may be reduced to the linear case, from which the equations of equilibrium referred to rectangular Cartesian coordinates follow. Similarly, the constitutive equations of Section 3 may be written in rectangular Cartesian coordinates by the use of (4.1).

Converting the kinematic and equilibrium equations into matrix form by using the matrix interpretations of the variables given in the previous section, it is observed that the field equations uncouple into those for generalized plane stress and those for transverse deflection. At this point it is considered instructive, and for ease of future use, to collect all pertinent equations into the previously mentioned two categories and to list them in the following way.

(a) *Extensional theory*

$$\begin{aligned} 2e_{\alpha\beta} &= U_{\alpha,\beta} + U_{\beta,\alpha}, \\ \bar{\bar{\boldsymbol{\delta}}}_i &= \bar{\boldsymbol{\delta}}_i, \\ \bar{\boldsymbol{\varkappa}}_{i\alpha} &= \bar{\bar{\boldsymbol{\delta}}}_{i,\alpha} = \bar{\boldsymbol{\delta}}_{i,\alpha}, \\ 2\bar{\boldsymbol{\varkappa}}_{(\alpha\beta)} &= \bar{\boldsymbol{\delta}}_{\alpha,\beta} + \bar{\boldsymbol{\delta}}_{\beta,\alpha}; \end{aligned} \tag{4.2}$$

$$N_{\beta\alpha,\alpha}+p_\beta = 0,$$
$$\tilde{\mathbf{M}}_{i\alpha,\alpha}+\tilde{\mathbf{l}}_i-\tilde{\mathbf{m}}_i = \mathbf{0}, \tag{4.3}$$
$$N_{\alpha\beta} = N_{\beta\alpha};$$

$$N_{\beta\alpha} = \alpha_1\delta_{\alpha\beta}e_{\gamma\gamma}+2\alpha_2e_{\alpha\beta}+\delta_{\alpha\beta}\boldsymbol{\beta}_9^T\tilde{\boldsymbol{\delta}}_3+\delta_{\alpha\beta}\boldsymbol{\beta}_{10}^T\,\tilde{\boldsymbol{\varkappa}}_{\gamma\gamma}+\;2\boldsymbol{\beta}_{11}^T\tilde{\boldsymbol{\varkappa}}_{(\alpha\beta)},$$
$$\tilde{\mathbf{m}}_\alpha = \boldsymbol{\beta}_3\tilde{\boldsymbol{\delta}}_\alpha+\boldsymbol{\beta}_{13}\tilde{\boldsymbol{\varkappa}}_{3\alpha},$$
$$\tilde{\mathbf{m}}_3 = \boldsymbol{\beta}_4\tilde{\boldsymbol{\delta}}_3+e_{\gamma\gamma}\boldsymbol{\beta}_9+\boldsymbol{\beta}_{12}^T\tilde{\boldsymbol{\varkappa}}_{\gamma\gamma}, \tag{4.4}$$
$$\tilde{\mathbf{M}}_{\alpha\beta} = \delta_{\alpha\beta}\boldsymbol{\beta}_5\tilde{\boldsymbol{\varkappa}}_{\gamma\gamma}+\boldsymbol{\beta}_6\tilde{\boldsymbol{\varkappa}}_{\alpha\beta}+\boldsymbol{\beta}_7\tilde{\boldsymbol{\varkappa}}_{\beta\alpha}+\delta_{\alpha\beta}e_{\gamma\gamma}\boldsymbol{\beta}_{10}+2e_{\alpha\beta}\boldsymbol{\beta}_{11}+\delta_{\alpha\beta}\boldsymbol{\beta}_{12}\tilde{\boldsymbol{\delta}}_3,$$
$$\tilde{\mathbf{M}}_{3\alpha} = \boldsymbol{\beta}_8\tilde{\boldsymbol{\varkappa}}_{3\alpha}+\boldsymbol{\beta}_{13}^T\tilde{\boldsymbol{\delta}}_\alpha.$$

(b) *Bending theory*

$$\bar{\boldsymbol{\delta}}_\alpha = \boldsymbol{\delta}_\alpha-U_{3,\alpha}\boldsymbol{\varepsilon}_1,$$
$$\bar{\boldsymbol{\delta}}_3 = \boldsymbol{\delta}_3,$$
$$\boldsymbol{\varkappa}_{\alpha\beta} = \boldsymbol{\varkappa}_{(\alpha\beta)}+\boldsymbol{\varkappa}_{[\alpha\beta]},$$
$$\boldsymbol{\varkappa}_{(\alpha\beta)} = \tfrac{1}{2}(\bar{\boldsymbol{\delta}}_{\alpha,\beta}+\bar{\boldsymbol{\delta}}_{\beta,\alpha}) = \tfrac{1}{2}(\boldsymbol{\delta}_{\alpha,\beta}+\boldsymbol{\delta}_{\beta,\alpha})-U_{3,\alpha\beta}\boldsymbol{\varepsilon}_1, \tag{4.5}$$
$$\boldsymbol{\varkappa}_{[\alpha\beta]} = \tfrac{1}{2}(\bar{\boldsymbol{\delta}}_{\alpha,\beta}-\bar{\boldsymbol{\delta}}_{\beta,\alpha}) = \tfrac{1}{2}(\boldsymbol{\delta}_{\alpha,\beta}-\boldsymbol{\delta}_{\beta,\alpha}),$$
$$\boldsymbol{\varkappa}_{3\alpha} = \boldsymbol{\delta}_{3,\alpha};$$

$$N_{3\alpha,\alpha}+p_3 = 0,$$
$$\mathbf{M}_{i\alpha,\alpha}+\mathbf{l}_i-\mathbf{m}_i = \mathbf{0}, \tag{4.6}$$
$$N_{3\alpha}-\mathbf{m}_\alpha^T\boldsymbol{\varepsilon}_1 = 0;$$

$$\mathbf{m}_\alpha = \boldsymbol{\alpha}_3\boldsymbol{\delta}_\alpha+\boldsymbol{\alpha}_{13}\boldsymbol{\varkappa}_{3\alpha},$$
$$\mathbf{m}_3 = \boldsymbol{\alpha}_4\boldsymbol{\delta}_3+\boldsymbol{\alpha}_{12}^T\boldsymbol{\varkappa}_{\gamma\gamma},$$
$$\mathbf{M}_{(\alpha\beta)} = \delta_{\alpha\beta}\boldsymbol{\alpha}_5\boldsymbol{\varkappa}_{\gamma\gamma}+[\boldsymbol{\alpha}_6+\boldsymbol{\alpha}_7]\boldsymbol{\varkappa}_{(\alpha\beta)}+\delta_{\alpha\beta}\boldsymbol{\alpha}_{12}\boldsymbol{\delta}_3, \tag{4.7}$$
$$\mathbf{M}_{[\alpha\beta]} = (\boldsymbol{\alpha}_6-\boldsymbol{\alpha}_7)\boldsymbol{\varkappa}_{[\alpha\beta]},$$
$$\mathbf{M}_{3\alpha} = \boldsymbol{\alpha}_8\boldsymbol{\varkappa}_{3\alpha}+\boldsymbol{\alpha}_{13}^T\boldsymbol{\delta}_\alpha;$$

where the notation $\boldsymbol{\varepsilon}_N$ has been incorporated to signify a $P/2\times 1$ matrix, every term of which is zero, except the $N$th, which is unity. Also in the above equations

$$\mathbf{M}_{\alpha\beta} = \mathbf{M}_{(\alpha\beta)}+\mathbf{M}_{[\alpha\beta]}, \tag{4.8}$$

where

$$\mathbf{M}_{(\alpha\beta)} = \tfrac{1}{2}(\mathbf{M}_{\alpha\beta}+\mathbf{M}_{\beta\alpha}), \qquad \mathbf{M}_{[\alpha\beta]} = \tfrac{1}{2}(\mathbf{M}_{\alpha\beta}-\mathbf{M}_{\beta\alpha}), \tag{4.9}$$

and $\tilde{\mathbf{l}}_i$ and $\mathbf{l}_i$ are surface moment $P/2\times 1$ matrices consistent with the above separation.

The three dimensional displacement field given by (2.35) with $B_\mu^\lambda = 0$ may then be calculated from

$$U_\alpha^* = U_\alpha+\boldsymbol{\zeta}^T\bar{\boldsymbol{\delta}}_\alpha+\boldsymbol{\eta}^T\tilde{\boldsymbol{\delta}}_\alpha,$$
$$U_3^* = U_3+\boldsymbol{\eta}^T\boldsymbol{\delta}_3+\boldsymbol{\zeta}^T\tilde{\boldsymbol{\delta}}_3. \tag{4.10}$$

where

$$\boldsymbol{\zeta}^T = (\xi, \xi^3, \xi^5, \ldots, \xi^{P-1}),$$
$$\boldsymbol{\eta}^T = (\xi^2, \xi^4, \xi^6, \ldots, \xi^P). \tag{4.11}$$

Again with the understanding discussed previously the three dimensional strains may be derived from (2.30)–(2.32) as

$$\begin{aligned} \gamma_{\alpha\beta} &= e_{\alpha\beta} + \boldsymbol{\zeta}^T \boldsymbol{\varkappa}_{(\alpha\beta)} + \boldsymbol{\eta}^T \tilde{\boldsymbol{\varkappa}}_{(\alpha\beta)}, \\ 2\gamma_{\alpha 3} &= \boldsymbol{\zeta}^T_{,3} \boldsymbol{\delta}_\alpha + \boldsymbol{\eta}^T \boldsymbol{\delta}_{3,\alpha} + \boldsymbol{\zeta}^T \bar{\bar{\boldsymbol{\delta}}}_{3,\alpha} + \boldsymbol{\eta}^T_{,3} \bar{\boldsymbol{\delta}}_\alpha, \\ \gamma_{33} &= \boldsymbol{\eta}^T_{,3} \boldsymbol{\delta}_3 + \boldsymbol{\zeta}^T_{,3} \bar{\bar{\boldsymbol{\delta}}}_3. \end{aligned} \tag{4.12}$$

All further considerations are restricted to the infinitesimal bending of an isotropic plate which is in equilibrium. In a manner similar to that of Green and Naghdi [9], the Stokes–Helmholtz decomposition theorem is applied, in its two dimensional form, by expressing $\boldsymbol{\delta}_\alpha$ as

$$\boldsymbol{\delta}_\alpha = \boldsymbol{\varphi}_{,\alpha} + \varepsilon_{\alpha\beta} \boldsymbol{\psi}_{,\beta}. \tag{4.13}$$

From (4.5) and (4.13) it follows that

$$\bar{\boldsymbol{\delta}}_\alpha = \boldsymbol{\chi}_{,\alpha} + \varepsilon_{\alpha\beta} \boldsymbol{\psi}_{,\beta}, \tag{4.14}$$

where

$$\boldsymbol{\chi} = \boldsymbol{\varphi} - U_3 \boldsymbol{\varepsilon}_1. \tag{4.15}$$

Substituting (4.14) into (4.5) then

$$\boldsymbol{\varkappa}_{(\alpha\beta)} = \boldsymbol{\chi}_{,\alpha\beta} + \tfrac{1}{2}(\varepsilon_{\alpha\gamma} \boldsymbol{\psi}_{,\gamma\beta} + \varepsilon_{\beta\gamma} \boldsymbol{\psi}_{,\gamma\alpha}), \tag{4.16}$$

$$\boldsymbol{\varkappa}_{[\alpha\beta]} = \tfrac{1}{2}(\varepsilon_{\alpha\gamma} \boldsymbol{\psi}_{,\gamma\beta} - \varepsilon_{\beta\gamma} \boldsymbol{\psi}_{,\gamma\alpha}) = \tfrac{1}{2} \varepsilon_{\alpha\beta} \Delta \boldsymbol{\psi}, \tag{4.17}$$

where "$\Delta$" = "$_{,\gamma\gamma}$" is the two dimensional Laplacian operator. Expressions for the constitutive equations in terms of the $P/2 \times 1$ matrix stress functions $\boldsymbol{\varphi}$, $\boldsymbol{\psi}$, $\boldsymbol{\chi}$, and the $P/2 \times 1$ matrix director displacement $\boldsymbol{\delta}_3$ are obtained by substituting (4.13), (4.16) and (4.17) into (4.7), thus

$$\mathbf{m}_\alpha = \boldsymbol{\alpha}_3(\boldsymbol{\varphi}_{,\alpha} + \varepsilon_{\alpha\beta} \boldsymbol{\psi}_{,\beta}) + \boldsymbol{\alpha}_{13} \boldsymbol{\delta}_{3,\alpha}, \tag{4.18}$$

$$\mathbf{m}_3 = \boldsymbol{\alpha}_4 \boldsymbol{\delta}_3 + \boldsymbol{\alpha}_{12}^T \Delta \boldsymbol{\chi}, \tag{4.19}$$

$$\mathbf{M}_{(\alpha\beta)} = \boldsymbol{\alpha}_5 \delta_{\alpha\beta} \Delta \boldsymbol{\chi} + \boldsymbol{\alpha}_{12} \delta_{\alpha\beta} \boldsymbol{\delta}_3 + (\boldsymbol{\alpha}_6 + \boldsymbol{\alpha}_7)\{\boldsymbol{\chi}_{,\alpha\beta} + \tfrac{1}{2}(\varepsilon_{\alpha\gamma} \boldsymbol{\psi}_{,\gamma\beta} + \varepsilon_{\beta\gamma} \boldsymbol{\psi}_{,\gamma\alpha})\}, \tag{4.20}$$

$$\mathbf{M}_{[\alpha\beta]} = \tfrac{1}{2} \varepsilon_{\alpha\beta} (\boldsymbol{\alpha}_6 - \boldsymbol{\alpha}_7) \Delta \boldsymbol{\psi}, \tag{4.21}$$

$$\mathbf{M}_{3\alpha} = \boldsymbol{\alpha}_8 \boldsymbol{\delta}_{3,\alpha} + \boldsymbol{\alpha}_{13}^T (\boldsymbol{\varphi}_{,\alpha} + \varepsilon_{\alpha\beta} \boldsymbol{\psi}_{,\beta}). \tag{4.22}$$

Before substituting these stress resultants into the equilibrium equations (4.6) the Stokes–Helmholtz decomposition is applied to $\mathbf{l}_\alpha$ so that

$$\mathbf{l}_\alpha = \mathbf{f}_{,\alpha} + \varepsilon_{\alpha\beta} \mathbf{g}_{,\beta}. \tag{4.23}$$

Substituting (4.20), (4.21), (4.23) and (4.18) into the equilibrium equation $(4.6)_2$, of the form

$$\mathbf{M}_{\alpha\beta,\beta} + \mathbf{l}_\alpha - \mathbf{m}_\alpha = \mathbf{0}, \tag{4.24}$$

it yields

$$\{\boldsymbol{\alpha}\Delta\boldsymbol{\chi}-\boldsymbol{\alpha}_3\boldsymbol{\varphi}+(\boldsymbol{\alpha}_{12}-\boldsymbol{\alpha}_{13})\boldsymbol{\delta}_3+\mathbf{f}\}_{,\alpha} = \varepsilon_{\alpha\gamma}\{\boldsymbol{\alpha}_3\boldsymbol{\psi}-\boldsymbol{\alpha}_6\Delta\boldsymbol{\psi}-\mathbf{g}\}_{,\gamma}, \tag{4.25}$$

where

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}_5+\boldsymbol{\alpha}_6+\boldsymbol{\alpha}_7. \tag{4.26}$$

Equations (4.25) are the Cauchy–Riemann conditions of the bracketed terms and hence these terms must be harmonic. Using a technique similar to that described in Green and Naghdi [9] the bracketed harmonic terms may, without loss in completeness, be taken to be equal to zero, that is

$$\boldsymbol{\alpha}\Delta\boldsymbol{\chi}-\boldsymbol{\alpha}_3\boldsymbol{\varphi}+(\boldsymbol{\alpha}_{12}-\boldsymbol{\alpha}_{13})\boldsymbol{\delta}_3+\mathbf{f} = \mathbf{0}, \tag{4.27}$$

$$\boldsymbol{\alpha}_6\Delta\boldsymbol{\psi}-\boldsymbol{\alpha}_3\boldsymbol{\psi}+\mathbf{g} = \mathbf{0}. \tag{4.28}$$

Upon substituting (4.22) and (4.19) into the equilibrium equation $(4.6)_2$, of the form

$$\mathbf{M}_{3\beta,\beta}+\mathbf{l}_3-\mathbf{m}_3 = \mathbf{0}, \tag{4.29}$$

it follows that

$$\boldsymbol{\alpha}_8\Delta\boldsymbol{\delta}_3-\boldsymbol{\alpha}_4\boldsymbol{\delta}_3+\boldsymbol{\alpha}_{13}^T\Delta\boldsymbol{\varphi}-\boldsymbol{\alpha}_{12}^T\Delta\boldsymbol{\chi}+\mathbf{l}_3 = \mathbf{0}, \tag{4.30}$$

while, by substituting $(4.6)_3$ into $(4.6)_1$ and utilizing (4.18), yields

$$\boldsymbol{\varepsilon}_1^T[\boldsymbol{\alpha}_3\Delta\boldsymbol{\varphi}+\boldsymbol{\alpha}_{13}\Delta\boldsymbol{\delta}_3]+p_3 = 0. \tag{4.31}$$

The basic equations governing $\boldsymbol{\varphi}$, $\boldsymbol{\chi}$, $\boldsymbol{\delta}_3$ and $\boldsymbol{\psi}$ are (4.27), (4.28), (4.30) and (4.31) and all other quantities are then determinable from (4.13) through (4.22).

Alternative forms of the basic equations may be obtained as follows. Eliminating $\boldsymbol{\chi}$ from between (4.27) and (4.30) it follows that

$$\{\boldsymbol{\alpha}_8\Delta-\boldsymbol{\alpha}_4+\boldsymbol{\alpha}_{12}^T\boldsymbol{\alpha}^{-1}(\boldsymbol{\alpha}_{12}-\boldsymbol{\alpha}_{13})\}\boldsymbol{\delta}_3+\{\boldsymbol{\alpha}_{13}^T\Delta-\boldsymbol{\alpha}_{12}^T\boldsymbol{\alpha}^{-1}\boldsymbol{\alpha}_3\}\boldsymbol{\varphi}+\boldsymbol{\alpha}_{12}^T\boldsymbol{\alpha}^{-1}\mathbf{f}+\mathbf{l}_3 = \mathbf{0}. \tag{4.32}$$

Since in (4.28) $\boldsymbol{\alpha}_3$ and $\boldsymbol{\alpha}_6$ are square, real, symmetric and $\boldsymbol{\alpha}_6$ is positive definite, then from standard reductions of simultaneous quadratic forms, e.g. Perlis [20], and with

$$\boldsymbol{\psi} = \mathbf{P}\boldsymbol{\Psi}, \qquad \mathbf{P}^T\boldsymbol{\alpha}_6\mathbf{P} = \mathbf{1}, \qquad \mathbf{P}^T\boldsymbol{\alpha}_3\mathbf{P} = \boldsymbol{\lambda}, \qquad \mathbf{G} = \mathbf{P}^T\mathbf{g}, \tag{4.33}$$

equation (4.28) may be written in the form

$$\Delta\boldsymbol{\Psi}-\boldsymbol{\lambda}\boldsymbol{\Psi}+\mathbf{G} = \mathbf{0}. \tag{4.34}$$

In (4.33) and (4.34), $\mathbf{1}$ is the identity matrix, $\mathbf{P}$ is a real nonsingular matrix independent of $\theta^1$ and $\theta^2$ and $\boldsymbol{\lambda}$ is dia.$\{\lambda_1, \ldots, \lambda_{P/2}\}$ where the $\lambda_N$ are the characteristic roots of the polynomial equation

$$\det(\lambda\boldsymbol{\alpha}_6-\boldsymbol{\alpha}_3) = 0. \tag{4.35}$$

Eliminating $\boldsymbol{\chi}$ from (4.15) and (4.27), results in

$$\Delta U_3\boldsymbol{\varepsilon}_1 = (\mathbf{1}\Delta-\boldsymbol{\alpha}^{-1}\boldsymbol{\alpha}_3)\boldsymbol{\varphi}+\boldsymbol{\alpha}^{-1}(\boldsymbol{\alpha}_{12}-\boldsymbol{\alpha}_{13})\boldsymbol{\delta}_3+\boldsymbol{\alpha}^{-1}\mathbf{f}. \tag{4.36}$$

Hence, the alternative set of basic equations is (4.31), (4.32), (4.34) and (4.36). This decomposition has been made without prior approximations in the constitutive equations, and thus either of the basic sets of equations represents an exact formulation of the bending theory of isotropic elastic Cosserat plates with $P$ directors.

For the bending theory alone, the displacements of equation (4.10) may be expressed in terms of the stress functions as

$$\begin{aligned} U_\alpha^* &= \boldsymbol{\zeta}^T(\boldsymbol{\varphi}_{,\alpha} - U_{3,\alpha}\boldsymbol{\varepsilon}_1 + \varepsilon_{\alpha\gamma}\boldsymbol{\psi}_{,\gamma}), \\ U_3^* &= U_3 + \boldsymbol{\eta}^T\boldsymbol{\delta}_3, \end{aligned} \tag{4.37}$$

and since the theory is linear the extensional part may be superimposed.

The three dimensional strains of equation (4.12) in a similar manner become

$$\begin{aligned} \gamma_{\alpha\beta} &= \boldsymbol{\zeta}^T\{\boldsymbol{\varphi}_{,\alpha\beta} - U_{3,\alpha\beta}\boldsymbol{\varepsilon}_1 + \tfrac{1}{2}(\varepsilon_{\alpha\gamma}\boldsymbol{\psi}_{,\gamma\beta} + \varepsilon_{\beta\gamma}\boldsymbol{\psi}_{,\gamma\alpha})\}, \\ 2\gamma_{\alpha 3} &= \boldsymbol{\zeta}_{,3}^T(\boldsymbol{\varphi}_{,\alpha} + \varepsilon_{\alpha\gamma}\boldsymbol{\psi}_{,\gamma}) + \boldsymbol{\eta}^T\boldsymbol{\delta}_{3,\alpha}, \\ \gamma_{33} &= \boldsymbol{\eta}_{,3}^T\boldsymbol{\delta}_3, \end{aligned} \tag{4.38}$$

for the bending theory.

## 5. COMPARISONS AND CONCLUSIONS

As an illustration in the use of this theory the pure bending of a flat plate of uniform thickness $h$, will now be analysed. Consider an elastic plate which is subjected to a stress distribution in a manner similar to that specified in Section 90, of Love [21]. Recalling that for a plane surface $\mathring{\mu}_i^{\cdot j} = \delta_i^j$, the stresses throughout the plate may be taken as

$$t^{11} = \sigma_{11} = E\alpha\xi, \qquad t^{22} = \sigma_{22} = E\beta\xi, \tag{5.1}$$

with all other stress components zero. In (5.1) $\alpha$ and $\beta$ are constants, $E$ is Young's modulus and the rectangular coordinates originating at the center of the plate are $x_1$, $x_2$ and $\xi$. By considering the resultants defined in Section 2, the only nonzero matrix resultants throughout the plate are the constant

$$\mathbf{M}_{11} = \alpha\boldsymbol{\gamma}, \qquad \mathbf{M}_{22} = \beta\boldsymbol{\gamma}, \tag{5.2}$$

where $\boldsymbol{\gamma}$ is a $P/2 \times 1$ matrix whose $N$th term corresponds to the $(2N-1)$th director and is given by

$$\boldsymbol{\gamma} = \sum_{N=1}^{P/2} \gamma_N \boldsymbol{\varepsilon}_N, \tag{5.3}$$

where

$$\gamma_N = \frac{2E}{2N+1}\left(\frac{h}{2}\right)^{2N+1}, \qquad N = 1, 2, \ldots, P/2, \tag{5.4}$$

It should be noted that for a complete description of the stress conditions in the plate all higher order moments must be included. However, since a discussion concerning the inverse of an infinite matrix will not be entered into, $P$ will be assumed even and finite throughout this discussion and subsequent results will be examined with $P$ tending to infinity.

Guided by elasticity theory and the basic reduced equations of either set (4.27), (4.28), (4.30) and (4.31) or the alternative set (4.31), (4.32), (4.34) and (4.36) of the bending theory, the following values for the stress functions pertinent to this problem are suggested:

$$\boldsymbol{\varphi} = 2(C_1+C_2)\boldsymbol{\alpha}_3^{-1}\{\boldsymbol{\alpha}-(\boldsymbol{\alpha}_{12}-\boldsymbol{\alpha}_{13})\boldsymbol{\alpha}_4^{-1}\boldsymbol{\alpha}_{12}^T\}\boldsymbol{\varepsilon}_1, \tag{5.5}$$

$$\boldsymbol{\psi} = \mathbf{P}\boldsymbol{\Psi} = \mathbf{0}, \tag{5.6}$$

$$\boldsymbol{\chi} = C_1X_1^2\boldsymbol{\varepsilon}_1+C_2X_2^2\boldsymbol{\varepsilon}_1+\boldsymbol{\varphi}, \tag{5.7}$$

$$\boldsymbol{\delta}_3 = -2(C_1+C_2)\boldsymbol{\alpha}_4^{-1}\boldsymbol{\alpha}_{12}^T\boldsymbol{\varepsilon}_1. \tag{5.8}$$

In (5.5), (5.7) and (5.8), the constants $C_1$ and $C_2$ are determined through stress boundary conditions, specified in (5.2), as

$$C_1\boldsymbol{\varepsilon}_1 = \tfrac{1}{2}(\mathbf{B}^{-1}\mathbf{A}-\mathbf{A}^{-1}\mathbf{B})^{-1}(\mathbf{B}^{-1}\boldsymbol{\gamma}\alpha-\mathbf{A}^{-1}\boldsymbol{\gamma}\beta), \tag{5.9}$$

$$C_2\boldsymbol{\varepsilon}_1 = \tfrac{1}{2}(\mathbf{B}^{-1}\mathbf{A}-\mathbf{A}^{-1}\mathbf{B})^{-1}(\mathbf{B}^{-1}\boldsymbol{\gamma}\beta-\mathbf{A}^{-1}\boldsymbol{\gamma}\alpha), \tag{5.10}$$

where

$$\mathbf{A} = \boldsymbol{\alpha}-\boldsymbol{\alpha}_{12}\boldsymbol{\alpha}_4^{-1}\boldsymbol{\alpha}_{12}^T, \tag{5.11}$$

$$\mathbf{B} = \boldsymbol{\alpha}_5-\boldsymbol{\alpha}_{12}\boldsymbol{\alpha}_4^{-1}\boldsymbol{\alpha}_{12}^T, \tag{5.12}$$

provided that $\mathbf{A}$ and $\mathbf{B}$ are nonsingular. Hence, upon utilizing (4.15) and substituting (5.5) and (5.7) into (4.37), the displacements become

$$U_\alpha^* = 2C_\alpha X_\alpha\boldsymbol{\zeta}^T\boldsymbol{\varepsilon}_1 \qquad \text{no sum on } \alpha, \tag{5.13}$$

$$U_3^* = -C_1X_1^2-C_2X_2^2-2(C_1+C_2)\boldsymbol{\eta}^T\boldsymbol{\alpha}_4^{-1}\boldsymbol{\alpha}_{12}^T\boldsymbol{\varepsilon}_1. \tag{5.14}$$

Since the theory under discussion has been constructed in a manner which suggests its equivalence to elasticity theory, provided that the Maclaurin expansion converges, it appears possible to compare this solution with its three dimensional counterpart in order to identify some of the material coefficients. Comparisons of single director Cosserat solutions with the corresponding results in the classical theory of elasticity have previously been made, for coefficients pertaining to the pure bending of a Cosserat plate in [9], for combined extension and bending coefficients in a circular cylindrical Cosserat shell in [7], and for some of the extensional theory coefficients in [5].

The displacements for a rectangular plate bent by couples have been given in Love [21], for example, as

$$U_1^E = (\alpha-\nu\beta)X_1\xi, \tag{5.15}$$

$$U_2^E = (\beta-\nu\alpha)X_2\xi, \tag{5.16}$$

$$U_3^E = -\tfrac{1}{2}(\alpha-\nu\beta)X_1^2-\tfrac{1}{2}(\beta-\nu\alpha)X_2^2-\tfrac{1}{2}\nu(\alpha+\beta)\xi^2, \tag{5.17}$$

where $\nu$ is Poisson's ratio and superscript $E$ denotes elastic displacements. Comparing (5.15) and (5.16) with (5.13) values for the constants $C_1$ and $C_2$ are

$$C_1 = \tfrac{1}{2}(\alpha-\nu\beta), \tag{5.18}$$

$$C_2 = \tfrac{1}{2}(\beta-\nu\alpha). \tag{5.19}$$

By substituting these interpretations into (5.14) and comparing the resulting equation with (5.17) it follows that

$$\boldsymbol{\alpha}_4^{-1}\boldsymbol{\alpha}_{12}^T\boldsymbol{\varepsilon}_1 = \frac{v}{2(1-v)}\boldsymbol{\varepsilon}_1. \tag{5.20}$$

By implementation of equations (5.9) through (5.12) and (5.18) through (5.20) it can be shown that

$$(\boldsymbol{\alpha}_6+\boldsymbol{\alpha}_7)\boldsymbol{\varepsilon}_1 = \frac{(1-v)}{(1-v^2)}\boldsymbol{\gamma}, \tag{5.21}$$

and

$$\left[\boldsymbol{\alpha}_5-\frac{v}{2(1-v)}\boldsymbol{\alpha}_{12}\right]\boldsymbol{\varepsilon}_1 = \frac{v}{(1-v^2)}\boldsymbol{\gamma}. \tag{5.22}$$

Hence, the identification of certain combinations of the material coefficient matrices involved in the bending theory has been established in (5.20), (5.21) and (5.22). It is of interest to note that the contraction in thickness of a plate under pure bending conditions is determined in this theory's solution by incorporating two nonzero directors in the displacement field while $P$ tending to infinity is required for a description of the stress field throughout the plate.

It should be noted that the first term in the matrix equations (5.21) and (5.22) is given by

$$\alpha_{6(1,1)}+\alpha_{7(1,1)} = (1-v)D, \tag{5.23}$$

and

$$\alpha_{5(1,1)}-\frac{v}{2(1-v)}\alpha_{12(1,1)} = vD, \tag{5.24}$$

respectively, where

$$D = \frac{1}{(1-v^2)}\gamma_1 = \frac{Eh^3}{12(1-v^2)}. \tag{5.25}$$

In the above $\alpha_{5(1,1)}$, $\alpha_{6(1,1)}$, $\alpha_{7(1,1)}$ and $\alpha_{12(1,1)}$ denote the elements in the first row and first column of the matrices $\boldsymbol{\alpha}_5, \boldsymbol{\alpha}_6, \boldsymbol{\alpha}_7$ and $\boldsymbol{\alpha}_{12}$ respectively. For the single director case given in [9] their expressions for the material coefficients are exact since in that theory no $\alpha_{12}$ exists; however, when viewed from the three dimensional theory of elasticity the theory of an elastic Cosserat plate is an approximation. In addition the values given in (5.23) and (5.24) suggest that a theory based on two directors would include an approximation to the contraction effect.

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**Абстракт**—На основе работы Грина и других авторов исследуется изотермическая инфинитезимальная теория изгиба изотропных поверхностей Коссера с директорамир. Матричная запись представляет зту теорию способом легко подвергающимся порстым преобразованиям. Ограничивая теорию начально плоскими поверхностями, она разделяется на часть удлинения и частв изгиба, Применяется к теории изгиба двухмерная матричная форма теоремы разложения Стокса-Гельгольца. Получаются при зтом четыре дифференциальные уравнения в уастных производных второго порядка в выражениях четырех функций напряжений. С помощью этих функций выражаются все кинематические переменные, остаточные напряжения и такие же напряжения высшего порядка. Эта теория применяется к чистому изгибу упругой пластинкм. Дается сравнение между хорошо сочетающейся терехмерной теорией упругости.